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# Harmonic oscillator with nonzero minimal uncertainties in both position and momentum in a SUSYQM framework

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## Abstract

In the context of a two-parameter  $(\alpha, \beta)$  deformation of the canonical commutation relation leading to nonzero minimal uncertainties in both position and momentum, the harmonic oscillator spectrum and eigenvectors are determined by using an extension of the techniques of conventional supersymmetric quantum mechanics (SUSYQM) combined with shape invariance under parameter scaling. The resulting supersymmetric partner Hamiltonians correspond to different masses and frequencies. The exponential spectrum is proved to reduce to a previously found quadratic spectrum whenever one of the parameters  $\alpha, \beta$  vanishes, in which case shape invariance under parameter translation occurs. In the special case where  $\alpha = \beta \neq 0$ , the oscillator Hamiltonian is shown to coincide with that of the  $q$ -deformed oscillator with  $q > 1$  and its eigenvectors are therefore  $n$ - $q$ -boson states. In the general case where  $0 \neq \alpha \neq \beta \neq 0$ , the eigenvectors are constructed as linear combinations of  $n$ - $q$ -boson states by resorting to a Bargmann representation of the latter and to  $q$ -differential calculus. They are finally expressed in terms of a  $q$ -exponential and little  $q$ -Jacobi polynomials.

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## 1. Introduction

Studies on small distances in string theory and quantum gravity suggest the existence of a finite lower bound to the possible resolution of length  $\Delta x_0$  (see, e.g., [1, 2]). On the other hand, on large scales there is no notion of plane waves or momentum eigenvectors on generic curved spaces. It has therefore been suggested that there could also exist a finite lower bound

to the possible resolution of momentum  $\Delta p_0$  (see, e.g., [3]). It is a natural, though nontrivial, assumption that minimal length and momentum should quantum theoretically be described as nonzero minimal uncertainties in position and momentum measurements.

Such nonzero minimal uncertainties can be described in the framework of small corrections to the canonical commutation relation [4, 5]

$$[x, p] = i\hbar(1 + \bar{\alpha}x^2 + \bar{\beta}p^2) \quad (1.1)$$

with  $\bar{\alpha} \geq 0$ ,  $\bar{\beta} \geq 0$ , and  $\bar{\alpha}\bar{\beta} < \hbar^{-2}$ . In such a context, they are given by  $\Delta x_0 = \hbar\sqrt{\bar{\beta}/(1 - \hbar^2\bar{\alpha}\bar{\beta})}$  and  $\Delta p_0 = \hbar\sqrt{\bar{\alpha}/(1 - \hbar^2\bar{\alpha}\bar{\beta})}$ , respectively<sup>3</sup>.

Since the canonical commutation relations lie in the very heart of quantum mechanics, studying the influence of small corrections to them in a quantum mechanical framework is interesting in its own right. It has indeed been argued [6] that such corrections may provide an effective description not only of strings but also of non-point-like particles such as quasiparticles and various collective excitations in solids, or composite particles such as nucleons and nuclei.

Solving quantum mechanical problems with the deformed canonical commutation relation (1.1) may, however, be a difficult task. In the special case where  $\bar{\alpha} = 0$  and  $\bar{\beta} > 0$ , the minimal uncertainty in the position turns out to be  $\Delta x_0 = \hbar\sqrt{\bar{\beta}}$ , whereas there is no nonzero minimal momentum uncertainty. As a consequence, equation (1.1) can be represented on momentum space wavefunctions (although not on position ones). Similarly, in the case where  $\bar{\alpha} > 0$  and  $\bar{\beta} = 0$ , there is only a nonzero minimal uncertainty in the momentum  $\Delta p_0 = \hbar\sqrt{\bar{\alpha}}$  and equation (1.1) can be represented on position space wavefunctions. In contrast, in the general case where  $\bar{\alpha} > 0$  and  $\bar{\beta} > 0$ , there is neither position nor momentum representation, so that one has to resort to a generalized Fock space representation or, equivalently, to the corresponding Bargmann representation [4, 5, 7].

The investigation of the harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \quad (1.2)$$

where  $x$  and  $p$  satisfy the deformed canonical commutation relation (1.1), is an interesting and nontrivial topic. The eigenvalue problem for such an oscillator

$$H|\psi_n\rangle = E_n|\psi_n\rangle \quad n = 0, 1, 2, \dots \quad (1.3)$$

has been solved exactly only in the case where  $\bar{\alpha} = 0$  and  $\bar{\beta} > 0$  by using the momentum representation and the technique of differential equations [8]. This approach has been recently extended to  $D$  dimensions [9] and some ladder operators have been constructed [10]. While it is obvious that in the case where  $\bar{\alpha} > 0$  and  $\bar{\beta} = 0$ , the eigenvalue problem can be treated in a similar way, that corresponding to both  $\bar{\alpha} > 0$  and  $\bar{\beta} > 0$  is more complicated and, as far as we know, has not been solved so far.

Over the years, it has been shown that supersymmetric quantum mechanics (SUSYQM) plays an important role in obtaining exact solutions of quantum mechanical problems (see, e.g., [11, 12]). In fact, all solvable quantum mechanical problems are either supersymmetric or can be made so.

Among the various exactly solvable potentials, there is a certain class of potentials characterized by a property known as shape invariance [13]. Shape invariant potentials are potentials such that their SUSY partner has the same spatial dependence with possibly altered parameters. For such potentials, both the energy eigenvalues and the wavefunctions can be

<sup>3</sup> In [4, 5], the two deforming parameters are denoted by  $\alpha$  and  $\beta$ . Here we reserve this notation for the dimensionless parameters to be introduced in section 2.

obtained algebraically without solving the differential equation [13, 14]. It turns out that the formalism of SUSYQM plus shape invariance (connected with translations of parameters) is intimately related to the factorization method developed by Schrödinger [15] and by Infeld and Hull [16] (for a comparison between these two methods and corresponding references see [12]). Other types of shape invariance, such as that connected with parameter scaling [17, 18], have provided a lot of new exactly solvable problems.

The purpose of the present paper is to extend SUSYQM and the notion of shape invariance to eigenvalue problems in the context of the deformed canonical commutation relation (1.1) and to apply such an extension to the case of the harmonic oscillator with nonzero minimal uncertainties in both position and momentum (i.e.,  $\bar{\alpha} > 0$  and  $\bar{\beta} > 0$ ). As a result, we will derive exact results for the energy spectrum and the eigenstates of such a system in a purely algebraic way.

This paper is organized as follows. The harmonic oscillator spectrum is obtained in section 2. Some special cases are reviewed in section 3. In section 4, the Hamiltonian eigenvectors are determined in explicit form. Section 5 contains the conclusion. Finally, some mathematical details are to be found in the appendix.

## 2. Harmonic oscillator spectrum in the general case

It is convenient to introduce dimensionless position and momentum operators,  $X = x/a$  and  $P = pa/\hbar$ , where  $a = \sqrt{\hbar/(m\omega)}$  is the oscillator characteristic length. They satisfy the commutation relation

$$[X, P] = i(1 + \alpha X^2 + \beta P^2) \tag{2.1}$$

where  $\alpha$  and  $\beta$  denote the dimensionless parameters  $\alpha = \bar{\alpha}\hbar/(m\omega)$  and  $\beta = \bar{\beta}m\hbar\omega$ , respectively. Here we have  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $\alpha\beta < 1$ .

The dimensionless harmonic oscillator Hamiltonian is then given by

$$h = \frac{H}{\hbar\omega} = \frac{1}{2}(P^2 + X^2) \tag{2.2}$$

and the corresponding eigenvalue problem reads

$$h|\psi_n\rangle = e_n|\psi_n\rangle \quad n = 0, 1, 2, \dots \tag{2.3}$$

where  $e_n = E_n/(\hbar\omega)$ . To find the spectrum  $e_n$ ,  $n = 0, 1, 2, \dots$ , we shall proceed in two steps: we will show that the Hamiltonian  $h$  is factorizable, then we will prove that the factorized Hamiltonian satisfies a condition similar to the shape invariance condition of conventional SUSYQM.

Let us first try to write  $h$  in the factorized form

$$h = B^+(g, s)B^-(g, s) + \epsilon_0 \tag{2.4}$$

where

$$B^\pm(g, s) = \frac{1}{\sqrt{2}}(sX \mp igP) \tag{2.5}$$

and  $\epsilon_0$  is the factorization energy. In (2.5),  $g$  and  $s$  are assumed to be two positive constants that are some functions of  $\alpha$ ,  $\beta$  and which go to 1 in the limit  $\alpha, \beta \rightarrow 0$ . The operators  $B^+(g, s)$  and  $B^-(g, s)$  are therefore counterparts of the standard harmonic oscillator creation and annihilation operators,  $a^+ = (X - iP)/\sqrt{2}$  and  $a = (X + iP)/\sqrt{2}$ , respectively.

Inserting (2.5) in (2.4) leads to the equation

$$h = \frac{1}{2}[(g^2 - \beta gs)P^2 + (s^2 - \alpha gs)X^2 - gs] + \epsilon_0. \tag{2.6}$$

For this expression of  $h$  to be equivalent to that given in (2.2), three conditions have to be fulfilled, namely

$$g^2 - \beta g s = 1 \quad (2.7)$$

$$s^2 - \alpha g s = 1 \quad (2.8)$$

$$\epsilon_0 = \frac{1}{2} g s. \quad (2.9)$$

It can be easily shown that equations (2.7) and (2.8) admit positive solutions for  $g$  and  $s$ , given by

$$g = sk \quad s = \frac{1}{\sqrt{1 - \alpha k}} \quad (2.10)$$

where

$$k \equiv \frac{1}{2}(\beta - \alpha) + \sqrt{1 + \frac{1}{4}(\beta - \alpha)^2}. \quad (2.11)$$

In deriving the expression of  $s$  in (2.10), use has been made of the equivalence between the conditions  $1 - \alpha k > 0$  and  $\alpha\beta < 1$ . Furthermore, one checks that  $k, g, s \rightarrow 1$  for  $\alpha, \beta \rightarrow 0$ , as it should be. We conclude that  $h$  can be written in form (2.4), where the factorization energy  $\epsilon_0$  is given in (2.9) and the two parameters  $\alpha, \beta$  of the problem have been replaced by their combinations  $g, s$ , defined in (2.10) and (2.11).

Let us now consider a hierarchy of Hamiltonians

$$h_i = B^+(g_i, s_i)B^-(g_i, s_i) + \sum_{j=0}^i \epsilon_j \quad i = 0, 1, 2, \dots \quad (2.12)$$

whose first member coincides with (2.4), i.e.,  $h_0 = h$ . Here  $g_i, s_i, \epsilon_i, i = 1, 2, \dots$  are assumed to be some positive parameters and  $g_0 = g, s_0 = s$ . We shall now proceed to prove that one can find values of  $g_i, s_i, \epsilon_i, i = 1, 2, \dots$ , such that the condition

$$B^-(g_i, s_i)B^+(g_i, s_i) = B^+(g_{i+1}, s_{i+1})B^-(g_{i+1}, s_{i+1}) + \epsilon_{i+1} \quad (2.13)$$

is satisfied. Equation (2.13), written in operator form, is exactly equivalent to the equation used in the factorization method [15, 16] (see also [19]) and is similar to the shape invariance condition of conventional SUSYQM [11–13]. Note that the factorization method (and thus the SUSY one) is quite general (see, e.g., [19]) and can be used for finding the eigenvalues of arbitrary Hermitian operators with a bounded-from-below spectrum. As it is obvious that the eigenvalues of the Hamiltonian (2.2) are positive, we can apply the factorization method or the SUSY one with shape invariance to our problem.

In explicit form, equation (2.13) reads

$$\begin{aligned} & \frac{1}{2}[(g_i^2 + \beta g_i s_i)P^2 + (s_i^2 + \alpha g_i s_i)X^2 + g_i s_i] \\ &= \frac{1}{2}[(g_{i+1}^2 - \beta g_{i+1} s_{i+1})P^2 + (s_{i+1}^2 - \alpha g_{i+1} s_{i+1})X^2 - g_{i+1} s_{i+1}] + \epsilon_{i+1} \end{aligned} \quad (2.14)$$

where  $i = 0, 1, 2, \dots$ . This leads to the set of three relations

$$g_{i+1}^2 - \beta g_{i+1} s_{i+1} = g_i^2 + \beta g_i s_i \quad (2.15)$$

$$s_{i+1}^2 - \alpha g_{i+1} s_{i+1} = s_i^2 + \alpha g_i s_i \quad (2.16)$$

$$\epsilon_{i+1} = \frac{1}{2}(g_i s_i + g_{i+1} s_{i+1}). \quad (2.17)$$

At this stage, it is worth noting that by multiplying (2.15) by  $\alpha$  and (2.16) by  $\beta$ , then subtracting, we get the equation

$$g_{i+1}^2 - \gamma^2 s_{i+1}^2 = g_i^2 - \gamma^2 s_i^2 \quad (2.18)$$

where

$$\gamma \equiv \sqrt{\frac{\beta}{\alpha}}. \tag{2.19}$$

On iterating (2.18) and using (2.7) and (2.8), it is then obvious that the recursion for  $g_i, s_i$  runs along the arc of the hyperbola  $g^2 - \gamma^2 s^2 = 1 - \gamma^2$  that lies in the first quadrant of the  $(g, s)$  plane. Since the equation of this hyperbola can also be written as  $uv = 1 - \gamma^2$ , where  $u \equiv g + \gamma s = 0$  and  $v \equiv g - \gamma s = 0$  are the equations of the asymptotes, it may prove useful to replace  $(g, s)$  by  $(u, v)$ .

Let us therefore introduce the new combinations of parameters

$$u_i = g_i + \gamma s_i \quad v_i = g_i - \gamma s_i. \tag{2.20}$$

The inverse transformation reads

$$g_i = \frac{1}{2}(u_i + v_i) \quad s_i = \frac{1}{2\gamma}(u_i - v_i) \tag{2.21}$$

where the assumptions  $g_i, s_i > 0$  impose that  $u_i > |v_i|$ . On substituting (2.21) into (2.15), (2.16), then combining the two resulting equations, we obtain the relations

$$u_{i+1}^2 + qv_{i+1}^2 = v_i^2 + qu_i^2 \tag{2.22}$$

$$u_{i+1}v_{i+1} = u_i v_i \tag{2.23}$$

where

$$q \equiv \frac{1 + \sqrt{\alpha\beta}}{1 - \sqrt{\alpha\beta}} > 1 \tag{2.24}$$

and equation (2.23) coincides with (2.18).

Equations (2.22) and (2.23) suggest a further transformation from  $u_i, v_i$  to

$$d_i = u_i v_i \quad t_i = \frac{v_i}{u_i} \tag{2.25}$$

where  $d_i$  and  $t_i$  have the same sign, which is that of  $v_i$ , and  $|t_i| < 1$ . According to (2.23),  $d_i$  is actually independent of  $i$ ,

$$d_i = d = uv \tag{2.26}$$

which amounts to the hyperbola equation previously obtained. On the other hand, equation (2.22) can be rewritten as

$$qt_{i+1} - t_i = \frac{qt_{i+1} - t_i}{t_i t_{i+1}} \tag{2.27}$$

thus showing that  $qt_{i+1} - t_i = 0$  or

$$t_i = q^{-i} t \quad t \equiv \frac{v}{u} = \frac{k - \gamma}{k + \gamma}. \tag{2.28}$$

From (2.25), (2.26) and (2.28), we also obtain

$$u_i = q^{i/2} u \quad v_i = q^{-i/2} v. \tag{2.29}$$

We conclude that the extended shape invariance condition (2.13) can indeed be satisfied by keeping the combination of parameters  $d$  constant while scaling the other combination of parameters  $t$  according to equation (2.28). Note that for  $i \rightarrow \infty, v_i \rightarrow 0$ , so that the recursion actually reaches the hyperbola asymptote lying in the first quadrant of the  $(g, s)$  plane.

The eigenvalues  $e_n$  of  $h$  are therefore given by

$$\begin{aligned} e_n(q, t) &= \sum_{i=0}^n \epsilon_i = \sum_{i=0}^{n-1} g_i s_i + \frac{1}{2} g_n s_n = \frac{1}{4\gamma} \left( \sum_{i=0}^{n-1} (u_i^2 - v_i^2) + \frac{1}{2} (u_n^2 - v_n^2) \right) \\ &= \frac{1}{4\gamma} \left\{ u^2 \left( [n]_q + \frac{1}{2} q^n \right) - v^2 \left( [n]_{q^{-1}} + \frac{1}{2} q^{-n} \right) \right\} \\ &= \frac{1}{4\gamma} \left\{ \left( u^2 - \frac{v^2}{q^{n-1}} \right) [n]_q + \frac{1}{2} \left( u^2 q^n - \frac{v^2}{q^n} \right) \right\} \\ &= \frac{u^2}{4\gamma} \left\{ \left( 1 - \frac{t^2}{q^{n-1}} \right) [n]_q + \frac{1}{2} \left( q^n - \frac{t^2}{q^n} \right) \right\} \end{aligned} \quad (2.30)$$

where we have successively used equations (2.9), (2.17), (2.21), (2.29), (2.28) and the definitions

$$[n]_q \equiv \frac{q^n - 1}{q - 1} \quad [n]_{q^{-1}} \equiv \frac{q^{-n} - 1}{q^{-1} - 1} = q^{-n+1} [n]_q. \quad (2.31)$$

In (2.30), we employ the notation  $e_n(q, t)$  to stress that, apart from a multiplicative constant  $u^2/(4\gamma)$ , the energy eigenvalues depend on the two parameters  $q$  and  $t$ , defined in (2.24) and (2.28), respectively.

It can be easily shown that  $q(\alpha, \beta) = q(\beta, \alpha)$ ,  $u^2(\alpha, \beta)/[4\gamma(\alpha, \beta)] = u^2(\beta, \alpha)/[4\gamma(\beta, \alpha)]$  and  $t(\alpha, \beta) = -t(\beta, \alpha)$ , where we explicitly write down the dependence on the deformation parameters  $\alpha, \beta$ . As a result, the eigenvalues  $e_n$  are symmetric under exchange of  $\alpha$  and  $\beta$ .

From (2.30), we obtain for the ground-state and excitation energies

$$e_0(q, t) = \frac{u^2}{8\gamma} (1 - t^2) \quad (2.32)$$

and

$$e_n(q, t) - e_0(q, t) = \frac{1}{2} K^2 \left( 1 - \frac{t^2}{q^n} \right) [n]_q \quad n = 1, 2, \dots \quad (2.33)$$

where

$$K \equiv u \sqrt{\frac{q+1}{4\gamma}}. \quad (2.34)$$

Since  $q > 1$  and  $t^2 < 1$ , the excitation energies grow exponentially to infinity when  $n \rightarrow \infty$ . Such a feature has already been encountered before in SUSYQM and shape invariance associated with parameter scaling [17].

On using (2.5), (2.21), (2.28) and (2.29), the Hamiltonians (2.12) of the SUSYQM hierarchy can be written as

$$h_i = \frac{1}{2} (a_i P^2 + b_i X^2) + c_i \quad i = 0, 1, 2, \dots \quad (2.35)$$

where

$$\begin{aligned} a_i &= \frac{u^2}{2(q+1)} (q^i + t) \left( 1 + \frac{t}{q^{i-1}} \right) \\ b_i &= \frac{u^2}{2\gamma^2(q+1)} (q^i - t) \left( 1 - \frac{t}{q^{i-1}} \right) \\ c_i &= \frac{u^2}{4\gamma} \left( 1 - \frac{t}{q^{i-1}} \right) [i]_q. \end{aligned} \quad (2.36)$$

For the supersymmetric partner of  $h = h_0$ , for instance, we obtain

$$h_1 = \frac{1}{1 - \alpha k} \left\{ \frac{1}{2} [1 + (2\beta - \alpha)k] P^2 + \frac{1}{2} (1 + \alpha k) X^2 + k \right\} \tag{2.37}$$

where  $k$  is defined in (2.11). Going back to variables with dimensions, we get

$$H_i \equiv \hbar\omega h_i = \frac{p^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x^2 + c_i \hbar\omega \tag{2.38}$$

with

$$m_i = \frac{m}{a_i} \quad \omega_i = \sqrt{a_i b_i} \omega. \tag{2.39}$$

We conclude that the harmonic oscillator with nonzero minimal uncertainties in both position and momentum belongs to a hierarchy of Hamiltonians of the same type but with different masses and frequencies. The change of the kinetic energy term, appearing naturally in addition to the usual modification of the potential energy one, is a new feature arising from the deformation of the canonical commutation relation. It is distinct from the supersymmetric generation of combined potential and effective-mass variations that may be effected for Hamiltonians with both a position-dependent effective mass and a position-dependent potential in the context of conventional SUSYQM [20].

### 3. Some special cases

In the present section, we will examine the two special cases where one of the parameters, e.g.,  $\alpha$ , vanishes or both parameters  $\alpha, \beta$  are equal.

#### 3.1. Limit $\alpha \rightarrow 0$

We plan to show that in the limit  $\alpha \rightarrow 0$ , the energy spectrum (2.30) reproduces the results previously obtained in the momentum representation [8, 9].

For small  $\alpha$  values, the parameter  $q$  behaves as

$$q \simeq 1 + 2\sqrt{\alpha\beta} + O(\alpha) \tag{3.1}$$

so that

$$q^n \simeq 1 + 2n\sqrt{\alpha\beta} + O(\alpha) \quad [n]_q \simeq n + O(\sqrt{\alpha}). \tag{3.2}$$

Furthermore,

$$\begin{aligned} \frac{1}{4\gamma} \left( u^2 - \frac{v^2}{q^{n-1}} \right) &\simeq gs + \frac{1}{2} \beta s^2 (n - 1) + O(\sqrt{\alpha}) \\ \frac{1}{8\gamma} \left( u^2 q^n - \frac{v^2}{q^n} \right) &\simeq \frac{1}{2} gs + \frac{1}{2} \beta s^2 n + O(\sqrt{\alpha}). \end{aligned} \tag{3.3}$$

Inserting such results in (2.30), we get

$$e_n \simeq gs \left( n + \frac{1}{2} \right) + \frac{1}{2} \beta s^2 n^2 + O(\sqrt{\alpha}) \tag{3.4}$$

where

$$g \simeq \frac{1}{2} \beta + \sqrt{1 + \frac{1}{4} \beta^2} + O(\alpha) \quad s \simeq 1 + O(\alpha). \tag{3.5}$$

In the limit  $\alpha \rightarrow 0$ , we therefore obtain

$$e_n(\beta) = \left( n + \frac{1}{2} \right) \sqrt{1 + \frac{1}{4} \beta^2} + \frac{1}{2} \beta \left( n^2 + n + \frac{1}{2} \right) \tag{3.6}$$



in agreement with [8, 9]. We conclude that in such a limit, the exponential spectrum of the general case reduces to a quadratic one.

It is worth noting that the spectrum corresponding to  $\alpha = 0$  may also be derived directly from SUSYQM and shape invariance without resorting to a limiting procedure. Going back to the factorization conditions (2.7)–(2.9) and setting  $\alpha = 0$  therein, we are only left with a single parameter  $g = \frac{1}{2}\beta + \sqrt{1 + \frac{1}{4}\beta^2}$  since  $s = 1$ . Among the shape invariance conditions (2.15)–(2.17), only the first and third ones survive, namely

$$g_{i+1}(g_{i+1} - \beta) = g_i(g_i + \beta) \quad \epsilon_{i+1} = \frac{1}{2}(g_i + g_{i+1}). \quad (3.7)$$

The solution of the former is  $g_{i+1} = g_i + \beta$ , thus giving  $g_i = g + i\beta$ , while the latter directly leads to the energy spectrum (3.6) by using the relation  $e_n(\beta) = \sum_{i=0}^n \epsilon_i$ .

This shows that the harmonic oscillator with only a nonzero minimal uncertainty in the position is shape invariant under translation of the parameter  $g$ . The Hamiltonians  $h_i$  of the SUSYQM hierarchy now reduce to

$$h_i = \frac{1}{2}[(1 + i^2\beta^2 + 2i\beta\sqrt{1 + \frac{1}{4}\beta^2})P^2 + X^2 + i(i\beta + 2\sqrt{1 + \frac{1}{4}\beta^2})] \quad i = 0, 1, 2, \dots \quad (3.8)$$

From (2.38) and (2.39), it follows that in this special case  $m_i\omega_i^2 = m\omega^2$ , so that only the kinetic energy term in the Hamiltonian  $H_i$  is changed.

### 3.2. Case $\alpha = \beta \neq 0$

There also occurs a simplification in the general formula (2.30) for the energy spectrum whenever the dimensionless parameters  $\alpha$  and  $\beta$  are equal, which means that the original parameters  $\bar{\alpha}$  and  $\bar{\beta}$  are related through  $\bar{\alpha} = m^2\omega^2\bar{\beta}$ . In such a case, we find

$$\gamma = k = 1 \quad g = s = \frac{1}{\sqrt{1 - \alpha}} \quad q = \frac{1 + \alpha}{1 - \alpha} \quad (3.9)$$

and thus

$$u = 2g \quad v = t = 0. \quad (3.10)$$

After some simple transformation, equation (2.30) can be written as

$$e_n(q) = \frac{1}{4}(q + 1)\{(q + 1)[n]_q + 1\} = \frac{1}{4}(q + 1)([n]_q + [n + 1]_q). \quad (3.11)$$

On the other hand, the Hamiltonians  $h_i$  of the SUSYQM hierarchy now become

$$h_i = \frac{1}{2}\{q^i(P^2 + X^2) + (q + 1)[i]_q\}. \quad (3.12)$$

From (2.38) and (2.39), it follows that in  $H_i$  both the mass and the frequency are scaled according to  $m_i = q^{-i}m$ ,  $\omega_i = q^i\omega$ .

The energy spectrum (3.11) is similar to that of the  $q$ -deformed harmonic oscillator

$$h_{\text{osc}} = \frac{1}{4}(q + 1)\{b, b^+\} \quad (3.13)$$

where  $b^+$  and  $b$  are  $q$ -deformed boson creation and annihilation operators satisfying the relation [21, 22]

$$bb^+ - qb^+b = I. \quad (3.14)$$

Such operators can indeed be used to construct  $n$ - $q$ -boson states

$$|n\rangle_q = \frac{(b^+)^n}{\sqrt{[n]_q!}}|0\rangle_q \quad n = 0, 1, 2, \dots \quad (3.15)$$

spanning an orthonormal basis of a deformed Fock space  $\mathcal{F}_q$ , i.e.,

$${}_q\langle n'|n\rangle_q = \delta_{n',n} \quad \sum_{n=0}^{\infty} |n\rangle_q \langle n| = I. \tag{3.16}$$

In (3.15),  $|0\rangle_q$  is the normalized vacuum state, i.e.,  $b|0\rangle_q = 0$  and  ${}_q\langle 0|0\rangle_q = 1$ , and the  $q$ -factorial

$$[n]_q! \equiv \begin{cases} 1 & \text{if } n = 0 \\ [n]_q [n-1]_q \cdots [1]_q & \text{if } n = 1, 2, \dots \end{cases} \tag{3.17}$$

is defined in terms of the  $q$ -numbers  $[n]_q$  introduced in (2.31). Since

$$b^+|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q \quad b|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q \tag{3.18}$$

the  $n$ - $q$ -boson states  $|n\rangle_q$  turn out to be the eigenvectors of  $h_{\text{osc}}$  with eigenvalues  $e_n(q)$ , i.e.,  $h_{\text{osc}}|n\rangle_q = e_n(q)|n\rangle_q, n = 0, 1, 2, \dots$

This means that in the  $\alpha = \beta \neq 0$  case, the harmonic oscillator Hamiltonian (2.2) with nonzero minimal uncertainties in position and momentum must be reducible to the  $q$ -deformed oscillator Hamiltonian (3.13). Such an assertion is easily proved by setting

$$X = \frac{1}{2}\sqrt{q+1}(b^+ + b) \quad P = \frac{i}{2}\sqrt{q+1}(b^+ - b) \tag{3.19}$$

or, conversely,

$$b^+ = \frac{1}{\sqrt{q+1}}(X - iP) \quad b = \frac{1}{\sqrt{q+1}}(X + iP). \tag{3.20}$$

It can indeed be shown from the commutation relation (2.1) that  $b^+$  and  $b$ , as defined in (3.20), fulfil the  $q$ -commutation relation (3.14) and are such that  $h = h_{\text{osc}}$ .

#### 4. Harmonic oscillator eigenvectors in the general case

To construct the eigenvectors of the harmonic oscillator Hamiltonian (2.2) in the case where neither  $\alpha$  nor  $\beta$  vanishes, one has to resort to a generalized Fock space representation [4, 5, 7], wherein

$$X = \frac{1}{2}\sqrt{\gamma(q+1)}(b^+ + b) \quad P = \frac{i}{2}\sqrt{\frac{q+1}{\gamma}}(b^+ - b) \tag{4.1}$$

are represented in terms of creation and annihilation operators

$$b^+ = \frac{1}{\sqrt{q+1}} \left( \frac{1}{\sqrt{\gamma}}X - i\sqrt{\gamma}P \right) \quad b = \frac{1}{\sqrt{q+1}} \left( \frac{1}{\sqrt{\gamma}}X + i\sqrt{\gamma}P \right) \tag{4.2}$$

satisfying the  $q$ -commutation relation (3.14). It is worth noting that in the special case where  $\alpha = \beta$ , considered in section 3.2, the operators (4.2) reduce to the operators (3.20) as a consequence of equation (3.9). In the general case where  $\alpha \neq \beta$ , however, the Hamiltonians  $h$  and  $h_{\text{osc}}$ , defined in (2.2) and (3.13), respectively, do not coincide any more since we have instead the relation

$$h = \frac{1}{8}(q+1) \left( \left( \gamma - \frac{1}{\gamma} \right) [(b^+)^2 + b^2] + \left( \gamma + \frac{1}{\gamma} \right) \{b, b^+\} \right). \tag{4.3}$$

Hence the eigenvectors of  $h$  are some linear combinations of the  $n$ - $q$ -boson states defined in (3.15).

SUSYQM and shape invariance provide us with some prescriptions to construct such linear combinations [11, 12, 14]. The ground state is indeed the normalized state annihilated by the operator  $B^-(g, s)$ , defined in (2.5), which we shall rewrite here as  $B^-(q, t)$ , i.e.,

$$B^-(q, t)|\psi_0(q, t)\rangle = 0 \quad \langle\psi_0(q, t)|\psi_0(q, t)\rangle = 1 \quad (4.4)$$

while the normalized excited states can be determined recursively through the equations

$$|\psi_{n+1}(q, t)\rangle = [e_{n+1}(q, t) - e_0(q, t)]^{-1/2} B^+(q, t)|\psi_n(q, t_1)\rangle \quad n = 0, 1, 2, \dots \quad (4.5)$$

where, according to (2.28),  $t_1 = t/q$ .

On using (2.21), (2.28), (2.29) and (4.1), the operators  $B^\pm(q, t)$  can be written as linear combinations of  $b^+$  and  $b$ ,

$$B^+(q, t) = \sqrt{\frac{q+1}{8\gamma}}(ub^+ - vb) = \frac{1}{\sqrt{2}}K(b^+ - tb) \quad (4.6)$$

$$B^-(q, t) = \sqrt{\frac{q+1}{8\gamma}}(ub - vb^+) = \frac{1}{\sqrt{2}}K(b - tb^+) \quad (4.7)$$

where  $K$  is given in (2.34).

To work out the explicit form of  $|\psi_n(q, t)\rangle$ ,  $n = 0, 1, 2, \dots$ , it is convenient to use the ( $q$ -deformed) Bargmann representation of the  $q$ -boson operators  $b^+$ ,  $b$ , associated with the corresponding  $q$ -deformed coherent states [22]. In such a representation the  $n$ - $q$ -boson states  $|n\rangle_q$  are represented by the functions

$$\varphi_n(q; \xi) = \frac{\xi^n}{\sqrt{[n]_q!}} \quad (4.8)$$

so that any vector  $|\psi\rangle_q = \sum_{n=0}^{\infty} c_n(q)|n\rangle_q \in \mathcal{F}_q$  is realized by the entire function  $\psi(q; \xi) = \sum_{n=0}^{\infty} c_n(q)\varphi_n(q; \xi)$ , belonging to a  $q$ -deformed Bargmann space  $\mathcal{B}_q$ , whose scalar product has been given in [22]. The operators  $b^+$  and  $b$  become the operator of multiplication by a complex number  $\xi$  and the  $q$ -differential operator  $\mathcal{D}_q$ , defined by

$$\mathcal{D}_q\psi(q; \xi) = \frac{\psi(q; q\xi) - \psi(q; \xi)}{(q-1)\xi} \quad (4.9)$$

respectively. Hence,  $B^\pm(q, t)$  are represented by

$$B^+(q, t) = \frac{1}{\sqrt{2}}K(\xi - t\mathcal{D}_q) \quad B^-(q, t) = \frac{1}{\sqrt{2}}K(\mathcal{D}_q - t\xi). \quad (4.10)$$

The first equation in (4.4) can therefore be rewritten as

$$(\mathcal{D}_q - t\xi)\psi_0(q, t; \xi) = 0. \quad (4.11)$$

As detailed in the appendix, the solution of this first-order  $q$ -difference equation can be easily obtained as

$$\psi_0(q, t; \xi) = \mathcal{N}_0(q, t)E_{q^2}\left(\frac{t}{q+1}\xi^2\right) \quad (4.12)$$

where the  $q$ -exponential  $E_q(\xi)$  is defined by [23]

$$E_q(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_q!} \quad (4.13)$$

and  $\mathcal{N}_0(q, t)$  is some normalization coefficient, determined by the second condition in equation (4.4).

From (4.8) and (4.13), it follows that the ground-state Bargmann wavefunction can also be written as

$$\psi_0(q, t; \xi) = \mathcal{N}_0(q, t) \sum_{\nu=0}^{\infty} \left( \frac{[2\nu - 1]_q!!}{[2\nu]_q!!} \right)^{1/2} t^\nu \varphi_{2\nu}(q; \xi) \tag{4.14}$$

where

$$[2\nu - 1]_q!! \equiv \begin{cases} 1 & \text{if } \nu = 0 \\ [2\nu - 1]_q [2\nu - 3]_q \cdots [1]_q & \text{if } \nu = 1, 2, \dots \end{cases} \tag{4.15}$$

and

$$[2\nu]_q!! \equiv \begin{cases} 1 & \text{if } \nu = 0 \\ [2\nu]_q [2\nu - 2]_q \cdots [2]_q & \text{if } \nu = 1, 2, \dots \end{cases} \tag{4.16}$$

The ground state is therefore a superposition of all even- $n$ - $q$ -boson states.

The orthonormality of the functions  $\varphi_n(q; \xi)$  in  $\mathcal{B}_q$  implies that the normalization coefficient in (4.12) and (4.14) is given by

$$\mathcal{N}_0(q, t) = \left( \sum_{\nu=0}^{\infty} \frac{[2\nu - 1]_q!!}{[2\nu]_q!!} t^{2\nu} \right)^{-1/2} = [{}_1\phi_0(q; -, q^2, t^2)]^{-1/2}. \tag{4.17}$$

Here  ${}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z)$  denotes a basic hypergeometric function (see [24] and the appendix). It can be easily checked that the series in (4.17) is convergent for all allowed  $t^2$  values, hence showing that the ground-state eigenvector is normalizable as it should be.

Considering next the excited states, we find that in Bargmann representation, the recursion relation (4.5) becomes the equation

$$\psi_{n+1}(q, t; \xi) = \left\{ [n + 1]_q \left( 1 - \frac{t^2}{q^{n+1}} \right) \right\}^{-1/2} (\xi - t\mathcal{D}_q) \psi_n(q, t; \xi) \tag{4.18}$$

where use has been made of equations (2.33) and (4.10). As shown in the appendix, the solution of this equation is given by

$$\psi_n(q, t; \xi) = \mathcal{N}_n(q, t) P_n(q, t; \xi) E_{q^2} \left( \frac{t}{(q + 1)q^n} \xi^2 \right). \tag{4.19}$$

Here  $P_n(q, t; \xi)$  denotes an  $n$ th-degree polynomial in  $\xi$ , satisfying the relation

$$P_{n+1}(q, t; \xi) = \xi P_n \left( q, \frac{t}{q}; \xi \right) - \xi \frac{t^2}{q^{n+1}} P_n \left( q, \frac{t}{q}; q\xi \right) - t\mathcal{D}_q P_n \left( q, \frac{t}{q}; \xi \right) \tag{4.20}$$

with  $P_0(q, t; \xi) \equiv 1$ , and  $\mathcal{N}_n(q, t)$  is a normalization coefficient fulfilling the recursion relation

$$\mathcal{N}_{n+1}(q, t) = \left\{ [n + 1]_q \left( 1 - \frac{t^2}{q^{n+1}} \right) \right\}^{-1/2} \mathcal{N}_n \left( q, \frac{t}{q} \right) \tag{4.21}$$

with  $\mathcal{N}_0(q, t)$  given in (4.17).

The solution of equation (4.21) is easily found to be given by

$$\mathcal{N}_n(q, t) = \{ [n]_q! (q^{-2n+1} t^2; q)_{n-1} \phi_0(q; -, q^2, q^{-2n} t^2) \}^{-1/2} \tag{4.22}$$

where the symbol  $(a; q)_n$  is defined in equation (A.5).

Solving equation (4.20) for the polynomials  $P_n(q, t; \xi)$  looks however more involved. First of all, let us remark that this equation may be considered as some  $q$ -difference analogue of a differential equation satisfied by Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x) \tag{4.23}$$

which can be obtained by combining equations (22.7.13) and (22.8.7) of [25]. As a matter of fact, in the formal limit where  $q \rightarrow 1$  while  $t^2$  remains different from one<sup>4</sup>, equation (4.20) can be rewritten as

$$Q_{n+1}(t; \xi) = (1 - t^2)\xi Q_n(t; \xi) - tQ'_n(t; \xi) \quad (4.24)$$

with  $Q_n(t; \xi) \equiv P_n(1, t; \xi)$ , because the  $q$ -differential operator  $\mathcal{D}_q$  goes to the ordinary differential operator  $d/d\xi$ . From (4.23), the solution of (4.24) is found to be

$$Q_n(t; \xi) = c^n(t)H_n[a(t)\xi] \quad a(t) = \sqrt{\frac{1-t^2}{2t}} \quad c(t) = \sqrt{\frac{1}{2}t(1-t^2)}. \quad (4.25)$$

We may therefore regard  $P_n(q, t; \xi)$  as some two-parameter deformation of Hermite polynomials. As far as we know, such a deformation has not been considered elsewhere. In the remainder of this section, we shall therefore devote ourselves to deriving an explicit solution to equation (4.20).

It is straightforward to show that for the first few  $n$  values, the polynomials  $P_n(q, t; \xi)$  are given by

$$P_1(q, t; \xi) = \left(1 - \frac{t^2}{q}\right)\xi \quad (4.26)$$

$$P_2(q, t; \xi) = \left(1 - \frac{t^2}{q^3}\right)\left[\left(1 - \frac{t^2}{q}\right)\xi^2 - t\right] \quad (4.27)$$

$$P_3(q, t; \xi) = \left(1 - \frac{t^2}{q^5}\right)\left(1 - \frac{t^2}{q^3}\right)\left[\left(1 - \frac{t^2}{q}\right)\xi^3 - \frac{t}{q}(1+q+q^2)\xi\right]. \quad (4.28)$$

For general  $n$  values, it is clear from the structure of equation (4.20) that

$$P_n(q, t; -\xi) = (-1)^n P_n(q, t; \xi) \quad (4.29)$$

which actually agrees with the examples shown in (4.26)–(4.28). We may therefore look for a solution of the type

$$P_n(q, t; \xi) = \sum_{m=0}^n \frac{1}{2} [1 + (-1)^{n-m}] f_{n,m}(q, t) \xi^m \quad (4.30)$$

where  $f_{n,m}(q, t)$ ,  $m = n, n-2, \dots, 0(1)$ , are some yet undetermined coefficients.

Inserting (4.30) in (4.20) and using the property  $\mathcal{D}_q \xi^m = [m]_q \xi^{m-1}$ , we get the recursion relations

$$f_{n+1,m}(q, t) = \left(1 - \frac{t^2}{q^{n-m+2}}\right) f_{n,m-1}\left(q, \frac{t}{q}\right) - t[m+1]_q f_{n,m+1}\left(q, \frac{t}{q}\right) \\ m = n-1, n-3, \dots, 0(1) \quad (4.31)$$

$$f_{n+1,n+1}(q, t) = \left(1 - \frac{t^2}{q}\right) f_{n,n}\left(q, \frac{t}{q}\right) \quad (4.32)$$

with  $f_{0,0}(q, t) \equiv 1$ . In (4.31), we have assumed that  $f_{n,-1}(q, t) = 0$  for odd values of  $n$ .

The solution of equation (4.32) is given by

$$f_{n,n}(q, t) = \prod_{k=0}^{n-1} \left(1 - \frac{t^2}{q^{2k+1}}\right) = \left(\frac{t^2}{q^{2n-1}}; q^2\right)_n \quad (4.33)$$

<sup>4</sup> By formal limit, we mean a limit that does not correspond to any physical values of the parameters  $\alpha, \beta$ . When  $q \rightarrow 1$ , we indeed obtain from (2.24) that either  $\alpha \rightarrow 0$  or  $\beta \rightarrow 0$ , which, from (2.11), (2.19) and (2.28), implies that either  $t \rightarrow -1$  or  $t \rightarrow 1$ .

with  $\prod_{k=0}^{-1} \equiv 1$  and  $(a; q^2)_n$  defined as in (A.5). As proved in the appendix, the solution of equation (4.31) is provided by

$$f_{n,m}(q, t) = \frac{[n]_q!}{[m]_q! [n-m]_q!!} \left( -\frac{t}{q^{(n+m-2)/2}} \right)^{(n-m)/2} \left( \frac{t^2}{q^{2n-1}}; q^2 \right)_{(n+m)/2} \tag{4.34}$$

where  $m = n - 2, n - 4, \dots, 0(1)$ . Comparison between equations (4.33) and (4.34) shows that the latter can be extended to all allowed  $m$  values, i.e.,  $m = n, n - 2, \dots, 0(1)$ . Note that the property  $f_{n,-1}(q, t) = 0$ , for odd  $n$  values, can be retrieved by using the usual assumption  $[n]_q! \rightarrow \infty$  if  $n \rightarrow -1$ .

Equations (4.30) and (4.34) therefore provide us with the general solution to equation (4.20), corresponding to  $P_0(q, t; \xi) = 1$ , and they include equations (4.26)–(4.28) as special cases.

To relate the polynomials  $P_n(q, t; \xi)$  with some known basic polynomials, let us first express them as basic hypergeometric functions [24]. Distinguishing between even and odd  $n$  values and using equations (A.4)–(A.7), we obtain

$$P_{2\nu}(q, t; \xi) = (q; q^2)_\nu \left( \frac{t^2}{q^{4\nu-1}}; q^2 \right)_\nu \left( \frac{t}{q^{\nu-1}(q-1)} \right)^\nu \times {}_2\phi_1 \left[ q^{-2\nu}, \frac{t^2}{q^{2\nu-1}}; q; q^2, -q^{2\nu}(q-1) \frac{\xi^2}{t} \right] \tag{4.35}$$

$$P_{2\nu+1}(q, t; \xi) = (q^3; q^2)_\nu \left( \frac{t^2}{q^{4\nu+1}}; q^2 \right)_{\nu+1} \left( \frac{t}{q^\nu(q-1)} \right)^\nu \xi \times {}_2\phi_1 \left[ q^{-2\nu}, \frac{t^2}{q^{2\nu-1}}; q^3; q^2, -q^{2\nu+1}(q-1) \frac{\xi^2}{t} \right] \tag{4.36}$$

where  $\nu = 0, 1, 2, \dots$ . Since in (4.35) and (4.36), the parameter  $a_1$  of the basic hypergeometric series is  $a_1 = q^{-2\nu}$ , where  $2\nu$  is a nonnegative integer, we deal here with terminating series.

From equations (4.35), (4.36), and the definition of little  $q$ -Jacobi polynomials [24]

$$p_n(x; a, b; q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx) \tag{4.37}$$

it is now obvious that the polynomials  $P_n(q, t; \xi)$  can be re-expressed in terms of the latter as

$$P_{2\nu}(q, t; \xi) = (q; q^2)_\nu \left( \frac{t^2}{q^{4\nu-1}}; q^2 \right)_\nu \left( \frac{t}{q^{\nu-1}(q-1)} \right)^\nu p_\nu \left[ -q^{2\nu-2}(q-1) \frac{\xi^2}{t}; \frac{1}{q}, \frac{t^2}{q^{4\nu}}; q^2 \right] \tag{4.38}$$

$$P_{2\nu+1}(q, t; \xi) = (q^3; q^2)_\nu \left( \frac{t^2}{q^{4\nu+1}}; q^2 \right)_{\nu+1} \left( \frac{t}{q^\nu(q-1)} \right)^\nu \xi \times p_\nu \left[ -q^{2\nu-1}(q-1) \frac{\xi^2}{t}; q, \frac{t^2}{q^{4\nu+2}}; q^2 \right]. \tag{4.39}$$

This completes the determination of the eigenfunctions (4.19) of the harmonic oscillator Hamiltonian (2.2) in ( $q$ -deformed) Bargmann representation. Combining equations (4.8), (4.12), (4.19) and (4.30), we can rewrite them as linear combinations of even- or odd- $n$ - $q$ -boson wavefunctions

$$\psi_{2\nu}(q, t; \xi) = \mathcal{N}_{2\nu}(q, t) \sum_{\sigma=0}^{\infty} \left[ \sum_{\mu=0}^{\min(\sigma, \nu)} \frac{\sqrt{[2\sigma]_q!}}{[2\sigma - 2\mu]_q!!} f_{2\nu, 2\mu}(q, t) \left( \frac{t}{q^{2\nu}} \right)^{\sigma-\mu} \right] \varphi_{2\sigma}(q, \xi) \tag{4.40}$$

$$\psi_{2\nu+1}(q, t; \xi) = \mathcal{N}_{2\nu+1}(q, t) \sum_{\sigma=0}^{\infty} \left[ \sum_{\mu=0}^{\min(\sigma, \nu)} \frac{\sqrt{[2\sigma+1]_q!}}{[2\sigma-2\mu]_q!!} f_{2\nu+1, 2\mu+1}(q, t) \left( \frac{t}{q^{2\nu+1}} \right)^{\sigma-\mu} \right] \times \varphi_{2\sigma+1}(q, \xi) \quad (4.41)$$

according to whether  $n$  is even or odd.

## 5. Conclusion

In the present paper, we have determined in a purely algebraic way both the spectrum and the eigenvectors of the harmonic oscillator with nonzero minimal uncertainties in both position and momentum by availing ourselves of an extension of SUSYQM and shape invariance powerful techniques to the case of the deformed canonical commutation relation (1.1).

In the present context, shape invariance is related to the scaling of some parameter  $t$ , depending in a complicated way upon the two deforming parameters  $\bar{\alpha}$ ,  $\bar{\beta}$  (or the two dimensionless ones  $\alpha$ ,  $\beta$ ) entering the canonical commutation relation. As occurs in other examples involving parameter scaling [17], the oscillator spectrum turns out to be exponential. The supersymmetric partner Hamiltonians correspond to both different masses and frequencies. Such an unusual feature is a direct consequence of the deformation of the commutation relation and is distinct from the combined potential and effective-mass variations that may be effected in the case of a position-dependent effective mass in the context of conventional SUSYQM [20].

We have proved that whenever one of the deforming parameters vanishes, e.g.,  $\alpha \rightarrow 0$  and  $\beta \neq 0$ , our exponential spectrum goes to the quadratic one, previously found by solving the deformed Schrödinger differential equation [8, 9]. Such a quadratic spectrum may also be derived from SUSYQM and shape invariance connected with parameter translation.

Furthermore, we have shown that when  $\alpha = \beta \neq 0$  or  $\bar{\alpha} = m^2 \omega^2 \bar{\beta} \neq 0$ , the harmonic oscillator with nonzero minimal uncertainties in both position and momentum reduces to the  $q$ -deformed harmonic oscillator corresponding to  $q > 1$  [22] and its eigenvectors therefore coincide with the  $n$ - $q$ -boson states with  $n = 0, 1, 2, \dots$

Finally, in the general case where  $0 \neq \alpha \neq \beta \neq 0$ , we have constructed the oscillator eigenvectors as linear combinations of  $n$ - $q$ -boson states by resorting to a ( $q$ -deformed) Bargmann representation of the latter and to  $q$ -differential calculus. The ground state can be expressed in terms of the  $q$ -exponential of an operator proportional to the square of the  $q$ -boson creation operator, acting on the vacuum, while the excited states contain as extra factors  $n$ th-degree polynomials in the creation operator. The latter are some two-parameter deformations of Hermite polynomials and can also be related to little  $q$ -Jacobi polynomials. It is worth noting that operators similar, but not identical, to that occurring in the ground-state eigenvector are familiar in other contexts, such as those of squeezed states [26] and of Bose–Einstein condensates [27].

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**Appendix. Some results used in the determination of the harmonic oscillator eigenvectors**

The  $q$ -exponential used in the present paper is defined by [23]

$$E_q(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_q!} \tag{A.1}$$

where the  $q$ -factorial  $[n]_q!$  is given in (3.17). For  $q > 1$  (see equation (2.24)), it converges for all finite values of  $\xi \in \mathbb{C}$ . It is the solution of the  $q$ -difference equation

$$\mathcal{D}_q E_q(a\xi) = a E_q(a\xi) \quad a \in \mathbb{C} \tag{A.2}$$

subject to the condition that  $E_q(0) = 1$ .

It follows from definition (4.9) and property (A.2) that

$$\begin{aligned} \mathcal{D}_q E_{q^2}(a\xi^2) &= \frac{E_{q^2}(aq^2\xi^2) - E_{q^2}(a\xi^2)}{(q-1)\xi} = a(q+1)\xi \frac{E_{q^2}(aq^2\xi^2) - E_{q^2}(a\xi^2)}{(q^2-1)a\xi^2} \\ &= a(q+1)\xi E_{q^2}(a\xi^2). \end{aligned} \tag{A.3}$$

Hence the function  $\psi_0(q, t; \xi)$ , defined in (4.12), satisfies the difference equation (4.11).

The basic hypergeometric function  ${}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z)$ , generalizing the conventional one  ${}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; z)$ , is defined by [24]

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \cdots (b_s; q)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n \end{aligned} \tag{A.4}$$

where  $z \in \mathbb{C}$  and

$$(a; q)_n \equiv \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}) & \text{if } n = 1, 2, \dots \end{cases} \tag{A.5}$$

In (A.4), it is assumed that the parameters  $b_1, b_2, \dots, b_s$  are such that the denominator factors in the terms of the series are never zero. Since  $(q^{-m}; q)_n = 0$  if  $n = m + 1, m + 2, \dots$ , an  ${}_r\phi_s$  series terminates if one of its numerator parameters is of the form  $q^{-m}$  with  $m = 0, 1, 2, \dots$ . When dealing with nonterminating series, it is assumed that the parameters and the variable are such that the series converges absolutely.

Some useful relations connecting  $(a; q)_n$  with the  $q$ -double factorials defined in (4.15) and (4.16), as well as the latter with  $q$ -factorials, are

$$[2v]_q!! = \frac{(q^2; q^2)_v}{(1-q)^v} \quad [2v-1]_q!! = \frac{(q; q^2)_v}{(1-q)^v} \quad [v]_{q^2}!! = \frac{[2v]_q!!}{(q+1)^v}. \tag{A.6}$$

We may also note the interesting relations [24]

$$(a; q)_{n+k} = (a; q)_n (aq^n; q)_k \quad (a^{-1}q^{1-n}; q)_n = (a; q)_n (-a^{-1})^n q^{-n(n-1)/2}. \tag{A.7}$$

Let us now prove that the functions  $\psi_n(q, t; \xi)$ , defined in (4.19), solve equation (4.18) provided conditions (4.20) and (4.21) are satisfied. For such a purpose, we shall apply the well-known rule for  $q$ -derivating a product of functions [23]

$$\mathcal{D}_q f(\xi)g(\xi) = f(q\xi)\mathcal{D}_q g(\xi) + g(\xi)\mathcal{D}_q f(\xi). \tag{A.8}$$



Substituting equation (4.19) into the right-hand side of (4.18) and using (A.8) and (A.3) successively, we obtain

$$\begin{aligned}
 \psi_{n+1}(q, t; \xi) &= \left\{ [n+1]_q \left( 1 - \frac{t^2}{q^{n+1}} \right) \right\}^{-1/2} \mathcal{N}_n(q, t_1) \left\{ \xi P_n(q, t_1; \xi) E_{q^2} \left( \frac{t_1}{(q+1)q^n} \xi^2 \right) \right. \\
 &\quad \left. - t P_n(q, t_1; q\xi) \mathcal{D}_q E_{q^2} \left( \frac{t_1}{(q+1)q^n} \xi^2 \right) - t E_{q^2} \left( \frac{t_1}{(q+1)q^n} \xi^2 \right) \mathcal{D}_q P_n(q, t_1; \xi) \right\} \\
 &= \left\{ [n+1]_q \left( 1 - \frac{t^2}{q^{n+1}} \right) \right\}^{-1/2} \mathcal{N}_n(q, t_1) \left\{ \xi P_n(q, t_1; \xi) - t P_n(q, t_1; q\xi) \frac{t_1}{q^n} \xi \right. \\
 &\quad \left. - t \mathcal{D}_q P_n(q, t_1; \xi) \right\} E_{q^2} \left( \frac{t_1}{(q+1)q^n} \xi^2 \right) \tag{A.9}
 \end{aligned}$$

which should coincide with equation (4.19) with  $n+1$  substituted for  $n$ . Comparison between the right-hand sides of both equations directly leads to (4.20) and (4.21), which completes the proof.

Let us finally consider the solution to the recursion relation (4.31). Substituting equation (4.34), where use is made of definition (A.5), into the right-hand side of (4.31), we get

$$\begin{aligned}
 f_{n+1,m}(q, t) &= \left( 1 - \frac{t^2}{q^{n-m+2}} \right) \frac{[n]_q!}{[m-1]_q! [n-m+1]_q!!} \left( -\frac{t}{q^{(n+m-1)/2}} \right)^{(n-m+1)/2} \\
 &\quad \times \prod_{k=(n-m+1)/2}^{n-1} \left( 1 - \frac{t^2}{q^{2k+3}} \right) - t [m+1]_q \frac{[n]_q!}{[m+1]_q! [n-m-1]_q!!} \\
 &\quad \times \left( -\frac{t}{q^{(n+m+1)/2}} \right)^{(n-m-1)/2} \prod_{k=(n-m-1)/2}^{n-1} \left( 1 - \frac{t^2}{q^{2k+3}} \right) \\
 &= \frac{[n]_q!}{[m]_q! [n-m+1]_q!!} ([m]_q + q^m [n-m+1]_q) \left( -\frac{t}{q^{(n+m-1)/2}} \right)^{(n-m+1)/2} \\
 &\quad \times \prod_{k=(n-m-1)/2}^{n-1} \left( 1 - \frac{t^2}{q^{2k+3}} \right) \\
 &= \frac{[n+1]_q!}{[m]_q! [n-m+1]_q!!} \left( -\frac{t}{q^{(n+m-1)/2}} \right)^{(n-m+1)/2} \left( \frac{t^2}{q^{2n+1}}; q^2 \right)_{(n+m+1)/2} \tag{A.10}
 \end{aligned}$$

where, in the last step, we employ the relation

$$[m]_q + q^m [n-m+1]_q = [n+1]_q \tag{A.11}$$

and equation (A.5) again. The final result in (A.10) coincides with equation (4.34) for  $n$  replaced by  $n+1$ . We conclude that equation (4.34) provides us with the solution to equation (4.31) corresponding to  $f_{0,0}(q, t) = 1$ .

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